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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A FRACTIONAL BOUNDARY VALUE PROBLEM WITH DIRICHLET BOUNDARY CONDITION

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Abstract. The authors consider a nonlinear fractional boundary value problem with the Dirichlet boundary condition. An associated Green's function is constructed as a series of functions by applying spectral theory. Criteria for the existence and uniqueness of solutions are obtained based on it.

1. Introduction

We study the boundary value problem (BVP) consisting of the fractional differential equation

$$-D^{\alpha}_{0+}u + a(t)u = w(t)f(t,u),\ 0 < t < 1, \tag{1.1}$$

and the Dirichlet boundary condition (BC)

$$u(0) = u(1) = 0, \tag{1.2}$$

where the following assumptions are satisfied:

(i) $1 < \alpha < 2$ and $a \in C[0,1]$,

(ii) $w \in L[0,1]$ such that $w(t) \not\equiv 0$ a.e. on $[0,1]$ and $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$,

(iii) $D^{\alpha}_{0+}h$ is the α-th Riemann–Liouville fractional derivative of h for $h : [0,1] \to \mathbb{R}$ defined by

$$D^{\alpha}_{0+}h(t) = \frac{1}{\Gamma(l-\alpha)} \frac{d^l}{dt^l} \int_0^t (t-s)^{l-\alpha-1} h(s)ds,\ l = \lfloor \alpha \rfloor + 1, \tag{1.3}$$

provided the right-hand side exists, where Γ is the Gamma function.

Fractional differential equations have extensive applications in various fields of science and engineering. Many phenomena in viscoelasticity, electrochemistry, control theory, porous media, electromagnetism, and other fields, can be

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modeled by fractional differential equations. We refer to the reader [10,14] and references therein for discussions of various applications.

The existence of solutions is an essential problem for BVPs involving fractional differential equations. This problem has been studied by many authors, for example, see [1–3,5–8,11,12,15,17] and references therein. As for integer order BVPs, Green's functions play an important role in the study of existence of solutions. However, due to the complexity of the fractional calculus, the Green's functions for fractional BVPs have not yet been well developed. In 2005, Bai and Lü [3] found that

$$G_0(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1 \end{cases} \tag{1.4}$$

is the Green's function for the BVP consisting of the equation

$$-D_{0+}^{\alpha} u = 0, \ 0 < t < 1, \tag{1.5}$$

and (1.2). This result was obtained by expressing the general solution of the equation

$$-D_{0+}^{\alpha} u = h(t)$$

in terms of the α-th Riemann–Liouville integral of h as defined by

$$I_{0+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds. \tag{1.6}$$

However, this method fails to work for the case when Eq. (1.5) is replaced by a more general equation

$$-D_{0+}^{\alpha} u + a(t)u = 0, \ 0 < t < 1, \tag{1.7}$$

due to the complexity caused by the extra term $a(t)u$. Recently, the present authors studied the problems consisting of Eq. (1.7) with $a(t) \equiv a$, a constant, and the BC

$$u(0) = 0, \ u(1) = aI_{0+}^{\alpha} u(1).$$

By using spectral theory, we derived the Green's function for this problem as a series of functions. By a similar approach, we also obtained the Green's function for the BVP

$$\begin{cases} -D_{0+}^{\alpha}u + aD_{0+}^{\gamma}u = 0, \ 0 < t < 1, \\ D_{0+}^{\beta}u(0) = 0, \ D_{0+}^{\alpha-\gamma}u(1) = au(1), \end{cases}$$

as a series of functions. We refer the reader to [7, Theorem 2.1] and [8, Theorem 2.1] for details. However, we would like to point out that there is a significant restriction in these two problems: the constant a in the second part of the BCs must be the same as the one in the equation. This unnatural assumption is required by technical arguments in the proofs.

In this paper, by applying spectral theory in a different way, we extend the Greens functions in the above problems to the BVP consisting of the equation (1.7) with the Dirichlet BC (1.2). We are then able to obtain results on the existence and uniqueness of solutions of BVP (1.1), (1.2). Our work provides a new approach for constructing Green's functions for fractional BVPs. This method can be further extended to BVPs with more general BCs.

This paper is organized as follows: After this introduction, our main results are stated in Section 2. Two examples are also given there. All the proofs are given in Section 3.

2. Main results

Throughout this paper, we assume

(H) There exists $\overline{a} > 0$ such that $|a(t)| \leq \overline{a} < 4^{\alpha-1}\Gamma(\alpha)$.

Define $G : [0,1] \times [0,1] \to \mathbb{R}$ by

$$G(t,s) = \sum_{n=0}^{\infty}(-1)^n G_n(t,s), \tag{2.1}$$

where G_0 is defined by (1.4) and

$$G_n(t,s) = \int_0^1 a(\tau)G_0(t,\tau)G_{n-1}(\tau,s)d\tau, \quad n \geq 1. \tag{2.2}$$

We then have the following result.

Theorem 2.1. *The function $G(t,s)$ defined by* (2.1) *as a series of functions is uniformly convergent for* $(t,s) \in [0,1] \times [0,1]$. *Furthermore, G is the Green's function for BVP* (1.7), (1.2).

With the Green's function G given in Theorem 2.1, we may apply some fixed point theorems to establish criteria for the existence of solutions of BVP (1.1), (1.2).

Define $\overline{G}: [0,1] \to \mathbb{R}$ by

$$\overline{G}(s) = \frac{\Gamma(\alpha) G_0(s,s)}{\Gamma(\alpha) - \overline{a} 4^{1-\alpha}}, \tag{2.3}$$

where $\overline{a}$ is defined in (H). Then we have the following theorem on the existence of a unique solution.

Theorem 2.2. *Assume f satisfies the Lipschitz condition in x*

$$|f(t,x_1) - f(t,x_2)| \le K|x_1 - x_2| \quad \text{for } (t,x_1),\ (t,x_2) \in [0,1] \times \mathbb{R},$$

with $K \in (0, 1/\int_0^1 \overline{G}(s) w(s) ds)$. Then BVP (1.1), (1.2) *has a unique solution. If, in addition, $f(t,0) \equiv 0$ on $[0,1]$, then BVP* (1.1), (1.2) *has no nontrivial solution.*

Theorem 2.3. *Assume*

$$\lim_{|x|\to\infty} \max_{t\in[0,1]} \frac{|f(t,x)|}{|x|} = 0 \tag{2.4}$$

and $f(t,0) \not\equiv 0$ on $[0,1]$. Then BVP (1.1), (1.2) *has at least one nontrivial solution.*

Remark 2.1. Condition (2.4) is satisfied by a broad range of functions. For instance, all bounded functions satisfy (2.4); unbounded functions such as $f(t,x) = p(t)x^q \operatorname{sgn} x + 1$ and $f(t,x) = p(t) + x^q \operatorname{sgn} x \ln(x^2+1) + e^{\sin x}$, $p \in C[0,1]$, $q \in (0,1)$, also satisfy (2.4).

To illustrate the application of our results, let us consider the following examples. We assume $\alpha \in (1,2)$ and $\overline{a}$ satisfies (H).

Example 1. Consider the BVP

$$\begin{cases} -D_{0+}^{\alpha}u + \overline{a}\cos(2\pi t)u = p\tan^{-1}u + e^t, \\ u(0) = u(1) = 0, \end{cases} \tag{2.5}$$

where $0 < p < 1/\int_0^1 \overline{G}(s)w(s)ds$. Let $f(t, x) = p\tan^{-1}x + e^t$. It is easy to see that $|f(t, x_1) - f(t, x_2)| \le p|x_1 - x_2|$ for any (t, x_1), $(t, x_2) \in [0, 1] \times \mathbb{R}$. Then by Theorem 2.2, BVP (2.5) has a unique solution. The solution is nontrivial since $f(t, 0) \not\equiv 0$.

Example 2. Consider the BVP

$$\begin{cases} -D_{0+}^{\alpha}u + \overline{a}\sin(2\pi t)u = \sqrt[3]{u} + \cos t, \\ u(0) = u(1) = 0. \end{cases} \tag{2.6}$$

Let $f(t, x) = \sqrt[3]{x} + \cos t$. Then, the conditions of Theorem 2.3 are satisfied, so BVP (2.6), has at least one nontrivial solution. Note that f does not satisfy the Lipschitz condition in x when x is near 0, and the solution may not be unique.

3. Proofs

The following lemma on the spectral theory in Banach spaces will be used to prove Theorem 2.1; see [16, page 795, items 57b and 57d] for details.

Lemma 3.1. *Let X be a Banach space, $\mathcal{A} : X \to X$ be a linear operator with the operator norm $\|\mathcal{A}\|$ and spectral radius $r(\mathcal{A})$ of $\mathcal{A}$. Then:*

(a) $r(\mathcal{A}) \le \|\mathcal{A}\|$;

(b) *if $r(\mathcal{A}) < 1$, then $(\mathcal{I} - \mathcal{A})^{-1}$ exists and $(\mathcal{I} - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} \mathcal{A}^n$, where $\mathcal{I}$ stands for the identity operator.*

The following lemma is excerpted from [3, Lemma 2.4].

Lemma 3.2. *Let G_0 be defined by (1.4). Then*

$$G_0(t, s) \le G_0(s, s) \le G_0(1/2, 1/2) = 4^{1-\alpha}/\Gamma(\alpha)$$

for $(t, s) \in [0, 1] \times [0, 1]$.

In the sequel, let $X = C[0,1]$ be the Banach space with the standard maximum norm.

Proof of Theorem 2.1. For any $h \in X$, let u be a solution of the BVP consisting of the equation

$$-D_{0+}^{\alpha} u + a(t)u = h(t),\ 0 < t < 1, \tag{3.1}$$

and BC (1.2). By (1.4),

$$u(t) = \int_0^1 G_0(t,s)(h(s) - a(s)u(s))ds,$$

or

$$u(t) + \int_0^1 a(s)G_0(t,s)u(s)ds = \int_0^1 G_0(t,s)h(s)ds. \tag{3.2}$$

Define $\mathcal{A}$ and $\mathcal{B}:\ X \to X$ by

$$(\mathcal{A}h)(t) = \int_0^1 G_0(t,s)h(s)ds, \tag{3.3}$$

$$(\mathcal{B}u)(t) = \int_0^1 a(s)G_0(t,s)u(s)ds. \tag{3.4}$$

Then (3.2) becomes

$$(\mathcal{I} + \mathcal{B})u = \mathcal{A}h. \tag{3.5}$$

In view of Lemma 3.2, it is easy to verify that $\|\mathcal{B}\| = \max_{\|u\|=1} \|\mathcal{B}u\| < 1$ when (H) holds. Hence, by Lemma 3.1, $r(\mathcal{B}) < 1$, and

$$u = \sum_{n=0}^{\infty} (-\mathcal{B})^n \mathcal{A}h. \tag{3.6}$$

We show that for $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$,

$$((-\mathcal{B})^n \mathcal{A}h)(t) = \int_0^1 (-1)^n G_n(t,s)h(s)ds. \tag{3.7}$$

Clearly, (3.7) holds for $n = 0$. Assume (3.7) holds for $n = m \geq 0$. Then by (2.2), (3.3), and (3.4),

$$\begin{aligned}
&((-\mathcal{B})^{m+1}\mathcal{A}h)(t) = (-\mathcal{B}(-\mathcal{B})^m\mathcal{A}h)(t) \\
&= \int_0^1 -a(\tau)G_0(t,\tau)\int_0^1 (-1)^m G_m(\tau,s)h(s)dsd\tau \\
&= \int_0^1 (-1)^{m+1}\int_0^1 a(\tau)G_0(t,\tau)G_m(\tau,s)d\tau h(s)ds \\
&= \int_0^1 (-1)^{m+1}G_{m+1}(t,s)h(s)ds,
\end{aligned}$$

i.e., (3.7) holds for $n = m+1$. By induction, (3.7) holds for any $n \in \mathbb{N}_0$.

We next show that for $n \in \mathbb{N}_0$,

$$|(-1)^n G_n(t,s)| \leq \frac{4^{(n+1)(1-\alpha)}\overline{a}^n}{\Gamma^{n+1}(\alpha)}, \tag{3.8}$$

where $\overline{a}$ is defined in (H). Clearly, (3.8) holds for $n = 0$. Assume (3.8) holds for $n = m \geq 0$. Then for any $(t,s) \in [0,1] \times [0,1]$,

$$\begin{aligned}
&|(-1)^{m+1}G_{m+1}(t,s)| \leq \int_0^1 |a(\tau)|G_0(t,\tau)G_m(\tau,s)d\tau \\
&\leq \int_0^1 \frac{4^{1-\alpha}\overline{a}}{\Gamma(\alpha)}G_m(\tau,s)d\tau \leq \int_0^1 \frac{4^{1-\alpha}\overline{a}}{\Gamma(\alpha)}\frac{4^{(m+1)(1-\alpha)}\overline{a}^m}{\Gamma^{m+1}(\alpha)}d\tau = \frac{4^{(m+2)(1-\alpha)}\overline{a}^{m+1}}{\Gamma^{m+2}(\alpha)},
\end{aligned}$$

i.e., (3.8) holds for $n = m+1$. By induction, (3.8) holds for any $n \in \mathbb{N}_0$.

By (H) we have $4^{1-\alpha}\overline{a}/\Gamma(\alpha) < 1$ on $[0,1]$. Hence by (2.1),

$$|G(t,s)| = \left|\sum_{n=0}^{\infty}(-1)^n G_n(t,s)\right| \leq \sum_{n=0}^{\infty}\frac{4^{(n+1)(1-\alpha)}\overline{a}^n}{\Gamma^{n+1}(\alpha)} < \infty \quad \text{on} \quad [0,1]\times[0,1].$$

Therefore, $G(t,s)$ as a series of functions is uniformly convergent on $[0,1]\times[0,1]$. By (2.1), (3.6), and (3.7),

$$u(t) = \sum_{n=0}^{\infty}\int_0^1 (-1)^n G_n(t,s)h(s)ds = \int_0^1 G(t,s)h(s)ds, \quad t \in [0,1]. \tag{3.9}$$

On the other hand, let u be defined by (3.9). By (2.1), (3.3), and (3.4), u satisfies (3.6). Hence, (3.5) holds. Again by (3.3) and (3.4), u satisfies (3.2). Therefore, u is a solution of BVP (3.1), (1.2). Thus, G is the Green's function for BVP (1.7), (1.2). $\square$

The next lemma is on the boundedness of the function G.

Lemma 3.3. *Let G and $\overline{G}$ be defined by* (2.1) *and* (2.3)*. Then for $(t,s) \in [0,1] \times [0,1]$, $|G(t,s)| \le \overline{G}(s)$.*

Proof. We first show that for $n \in \mathbb{N}_0$

$$|(-1)^n G_n(t,s)| \le G_0(s,s) \frac{4^{n(1-\alpha)} \overline{a}^n}{\Gamma^n(\alpha)}. \tag{3.10}$$

In fact, when $n = 0$, this follows from Lemma 3.2. Assume (3.10) holds for $n = m$. Then by (2.2), for $(t,s) \in [0,1] \times [0,1]$,

$$|(-1)^{m+1} G_{m+1}(t,s)| \le \int_0^1 |a(\tau)| G_0(t,\tau) |G_m(\tau,s)| d\tau$$
$$\le \int_0^1 \overline{a} G_0(\tau,\tau) G_0(s,s) \frac{4^{m(1-\alpha)} \overline{a}^m}{\Gamma^m(\alpha)} d\tau \le G_0(s,s) \frac{4^{(m+1)(1-\alpha)} \overline{a}^{m+1}}{\Gamma^{m+1}(\alpha)},$$

i.e., (3.10) holds for $n = m+1$. By induction, (3.10) holds for any $n \in \mathbb{N}_0$.

Combining (2.1), (2.3), and (3.10), we see that

$$|G(t,s)| = \left| \sum_{n=0}^{\infty} (-1)^n G_n(t,s) \right| \le G_0(s,s) \sum_{n=0}^{\infty} \frac{4^{n(1-\alpha)} \overline{a}^n}{\Gamma^n(\alpha)} = \overline{G}(s).$$

□

Now we prove Theorems 2.2 using the contraction mapping principle.

Proof of Theorem 2.2. Define $T: X \to X$ by

$$(Tu)(t) = \int_0^1 G(t,s) w(s) f(s, u(s)) ds, \ u \in X. \tag{3.11}$$

Clearly, T is completely continuous and $u(t)$ is a solution of BVP (1.1), (1.2) if and only if u is a fixed point of T in X.

For any u_1, $u_2 \in X$, and $t \in [0,1]$,

$$|(Tu_1 - Tu_2)(t)| = \left| \int_0^1 G(t,s)w(s)\left(f(s,u_1(s)) - f(s,u_2(s))\right) ds \right|$$
$$\leq \int_0^1 \overline{G}(s)w(s)\left|f(s,u_1(s)) - f(s,u_2(s))\right| ds$$
$$\leq \int_0^1 \overline{G}(s)w(s)K\left|u_1(s) - u_2(s)\right| ds$$
$$\leq \left(K \int_0^1 \overline{G}(s)w(s)ds \right) \|u_1 - u_2\|.$$

Note that $K \int_0^1 \overline{G}(s)w(s)ds < 1$. Hence, T is a contraction mapping. By the contraction mapping principle, T has a unique fixed point. Thus, BVP (1.1), (1.2) has a unique solution.

If, in addition, $f(t,0) \equiv 0$ on $[0,1]$, then obviously $u(t) \equiv 0$ is a solution of BVP (1.1), (1.2). By the uniqueness of solutions, BVP (1.1), (1.2) has no nontrivial solutions. □

Finally, we use the Schauder fixed point theorem to prove Theorem 2.3.

Proof of Theorem 2.3. Let $k = 1/\int_0^1 \overline{G}(s)w(s)ds$. Since

$$\lim_{|x|\to\infty} \max_{t\in[0,1]} \frac{|f(t,x)|}{|x|} = 0,$$

there exists $M_1 > 0$ such that $|f(t,x)| \leq k|x|$ for any $t \in [0,1]$ and x with $|x| \geq M_1$. Now $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$ implies there exists $F > 0$ such that $|f(t,x)| \leq F$ on $[0,1] \times [-M_1, M_1]$. Let $M_2 = \max\{M_1,\ F/k\}$. Then

$$|f(t,x)| \leq kM_2 \quad \text{on} \quad [0,1] \times [-M_2, M_2]. \tag{3.12}$$

Let $\Omega = \{u \in X \mid \|u\| \leq M_2\}$. For any $u \in \Omega$, $|u(t)| \leq M_2$ on $[0,1]$. By (3.11), (3.12), and Lemma 3.3, for $t \in [0,1]$,

$$|(Tu)(t)| = \left| \int_0^1 G(t,s)w(s)f(s,u(s))ds \right| \leq \int_0^1 |G(t,s)|w(s)|f(s,u(s))|ds$$
$$\leq kM_2 \int_0^1 \overline{G}(s)w(s)ds = M_2.$$

Hence, $\|Tu\| \le M_2$, i.e., $T(\Omega) \subset \Omega$. By the Schauder fixed point theorem, T has at least one fixed point in Ω. Clearly $u(t) \equiv 0$ is not a fixed point. Therefore, BVP (1.1), (1.2) has at least one nontrivial solution. $\square$

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